Light Escapes the Enchanted Forest

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October 19, 2023

1 Introduction

Jonas Pach posed the following "enchanted forest problem"[1, pg. 19]: Is it possible to trap all the light from a point source with circular mirrors? In this note I prove that the answer is *no*, and in fact, that a light ray in a random direction escapes with probability 1. The proof is a simple application of the ergodic theory of billiards. In this note I will formally define the problem, review the theorems and definitions needed from the ergodic theory of billiards, and present the proof.

Let's define the problem we are attempting to solve more precisely. Let $s \in \mathbb{R}^2$ be a point and $\{D_i\}$ be a finite disjoint collection of convex subsets of $\mathbb{R}^2$, with each D_i having a smooth (C^l, $l \geq 3$) compact boundary. Let $D = \cup_i D_i$, and $\mathcal{D} = \mathbb{R}^2 \setminus D$ be the domain in which light rays will move. Now define the *configuration space* of light rays in this domain as $\mathcal{D} \times S^1$(with S^1 being the space of unit vectors $\{v \in \mathbb{R}^2 \mid ||v|| = 1\}$), which we can think of as the location and direction of a given ray. The light rays evolve in time in the following way: if there are no collisions with the domain walls in a time interval t, the ray (x, y, v) evolves to $(x + tv_x, y + tv_y, v)$. When a ray collides with a domain wall, its velocity vector changes according to $v' = v - 2\langle v, n\rangle n$, with n being the normal to the boundary wall at the point of collision. Given an initial condition (x_0, y_0, v_0), these time evolution rules define a forward and backward time evolution $\Phi^t(x_0, y_0, v_0)$ for all $t \in \mathbb{R}$.

The problem we are considering is whether it is possible to trap all the light rays emanating from a single point s, that is, if it is possible that $\Phi^t(s_x, s_y, v)$ is bounded for all $t \geq 0$ and $v \in S^1$.

The answer is *no*, it is not possible. In fact, we will prove the following more general theorem:

Theorem 1 *For any given domain $D \subset \mathbb{R}^2$ of the sort described above, and any given $s \in D$, (s, v) eventually escapes for all $v \in S^1$ but a set of measure zero.*

To prove this, I first must introduce some notions from ergodic theory.

2 Ergodic Theory

The ergodic theory of billiards concerns the dynamics of idealized "billiards" reflecting off of obstacles on the torus or a bounded subset of the plane. "Billiards" are formally identical to "light rays" as defined in the previous section, so I will just refer to "light rays" when quoting definitions to reduce confusion.

We will consider light rays moving on a subset $\mathcal{D}$ of the torus $Tor^2 = [0,1]^2/\sim$, with $\sim$ being the equivalence relation identifying opposite sides of $[0,1]^2$. As in the previous section, $\mathcal{D}$ is defined as $Tor^2 \setminus D$, with D being a finite collection of convex subsets of $[0,1]^2$ with smooth compact boundary. The dynamics of the light rays are the same as in the previous section, with light rays moving to the opposite side of $[0,1]^2$ when colliding with an edge of the square.

To simplify the dynamics, Let $\mathcal{M} = \partial D \times [0, \pi]$ be the *collision space* of the domain, representing the position and angle (r, ϕ) of a light ray that has just collided with an obstacle. I will adopt the convention that the angle ϕ represents the *post*-collisional angle of the ray. We can now reduce the dynamics to a function $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ which maps from a position and post-collision angle, to the position and angle at the next collision the in the ray's trajectory.

Now I will introduce some terminology and theorems from [2]. A curve on $\mathcal{M}$ is *unstable*, if $\mathcal{K} \leq d\phi/dr \leq \mathcal{K} + \tau_{-1}$, where $\mathcal{K}$ is the curvature of the boundary at a given r and τ_{-1} is the length of the path preceding the collision at a given point. Similarly a *stable* curve satisfies $-\mathcal{K} - cos\phi/\tau \leq d\phi/dr \leq -\mathcal{K}$. [2, Sec. 4.5] Stable and unstable curves in the collision space correspond to wavefronts in the phase space; the condition $d\phi/dr \geq \mathcal{K}$ holds iff the wavefront is expanding, while $d\phi/dr \leq \mathcal{K} + \tau_{-1}$ holds iff the wavefront does not focus(converge to a single point) during its trajectory from its last point of collision.

A *homogeneity strip*[Sec. 5.3][2] is a region in a

$$\mathbb{H}_k = \{(r, \phi) : \pi/2 - k^{-2} < \phi < \pi/2 - (k+1)^{-2}\}$$

A *weakly homogeneous unstable curve* is an unstable curve lying entirely in a single homogeneity strip.[2, Sec. 5.3] A *homogeneous unstable curve* is a weakly homogeneous unstable curve satisfying certain bounds on their curvature and distortion under the map $\mathcal{F}$[3, Sec. 4.3]; all sufficiently smooth weakly homogeneous unstable curves satisfy these bounds[3, Appendix C]. *Homogeneous stable curves* are defined analogously.

The main theorem we will need is a special case of Theorem 7.31 of [2]. A Hölder continuous function is a function satisfying

$$|f(x) - f(y)| \leq C_f |x - y|^{\alpha_f}$$

for some constants $C_f > 0$ and $\alpha_f \in (0, 1]$.

Theorem 2 *Let W be a homogeneous unstable curve. Let μ be the Lebesgue measure on $\mathcal{M}$ and let ν_W be the uniform probability measure on W proportional to the Lebesgue measure induced by length along the curve: $\nu_W = \frac{1}{|W|}\mu_W$. For any Hölder continuous function f on $\mathcal{M}$ and any $n \geq 0$, we have*

$$\Big| \int_W f \circ \mathcal{F}^n d\nu_W - \int_{\mathcal{M}} f d\mu \Big| \leq B_f \theta_f^n$$

where $B_f > 0$ and $\theta_f \in [0, 1)$ are constants.

3 Proof

Now let us turn to the proof of Theorem 1.

First, we must translate the problem from $\mathbb{R}^2$ to Tor^2 so the theory of section 2 can be applied. To that end, given a set of obstacles $D = D_i \subset \mathbb{R}^2$ and a point $s \in \mathbb{R}^2$, we define sets $D' = D'_i \subset Tor^2$ and $s' \in Tor^2$ as follows: let $M \subset \mathbb{R}^2$ be a square of side length m which contains s and D_i, and identify M with Tor^2 by identifying opposite sides; let s' and D'_i be the corresponding subsets of Tor^2. Let B be the subset of Tor^2 corresponding to the sides of the square M. It is clear that a light ray (s, v) escapes to infinity in $\mathcal{D}$ iff the corresponding ray (s', v) in $\mathcal{D}' = Tor^2 \setminus D'$ hits B.

Consider the set of light rays that do *not* hit the set B after two bounces. In the collision space, after two bounces, these rays form a set of at most

n^2 unstable curves $W_i \subset \mathcal{M}$, n being the number of obstacles in D'. Since the map Φ is smooth away from collisions, the Lebesgue measure on W_i is absolutely continuous with respect to the Lebesgue measure on our set of initial conditions $(s', v), v \in S^1$. Thus if we can prove that all but a measure 0 subset of the rays in W_i eventually hit B, theorem 1 will be established. Let $B_{\mathcal{M}}$ be the positive-measure open subset of $\mathcal{M}$ consisting of light rays whose trajectory intersects B before their next collision.

Assume for the sake of contradiction that there is a positive measure subset of $\cup_i W_i$ which never hits B. Then there must be some particular W_j which has a positive measure subset which never hits B. W_j can in turn be divided into a countable number of homogeneous unstable curves W_j^k, and there must be some particular W_j^k which contains a positive measure set V which never hits B.

Since V is measurable, for any $\epsilon > 0$ we can approximate it up to a set of measure ϵ by a finite union of intervals $\cup_i I_i$ such that $\mu_W(V \Delta \cup_i I_i) < \epsilon$ Since these intervals are subsets of an unstable curve and lie in a single homogeneity strip, they are all unstable homogeneous curves. There must be at least one such curve I^* with $\mu_W(V \cap I^*) \geq (1-\epsilon)|I^*|$. Let ν_{I^*} be the uniform probability measure on $I*$.

Now consider some Hölder continuous function g with support in the interior of $B_{\mathcal{M}}$ with maximum M and total integral $T = \int_{\mathcal{M}} g d\mu > 0$. Now, since all but a subset of measure ϵ of V never intersects $B_{\mathcal{M}}$, we have for all n that $\int_{I^*} g \circ \mathcal{F}^n d\nu_{I^*} < M\epsilon$. Since ϵ can be made arbitrarily small, this integral can be made arbitrarily close to zero for all n by choice of I_i; on the other hand, by Theorem 2, by increasing n, we can make it arbitrarily close to $T > 0$, a contradiction. Therefore, the subset of $\cup_i W_i$ that never hits B is of measure zero. This concludes the proof of Theorem 1.

References

[1] Croft, Hallard T., Kenneth Falconer, and Richard K. Guy. *Unsolved problems in geometry: unsolved problems in intuitive mathematics.* Vol. 2. Springer Science and Business Media, 2012.

[2] Chernov, Nikolai, and Roberto Markarian. *Chaotic billiards.* No. 127. American Mathematical Soc., 2006.

[3] Chernov, Nikolai, and Dmitry Dolgopyat. *Brownian Brownian Motion-I*. American Mathematical Soc., 2009.